JOURNAL OF APPROXIMATION THEORY 14, 191–203 (1975)

On Bounding Spline Interpolation*

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DEFINITIONS

For given positive integers n and k, and for a given real nondecreasing sequence $\mathbf{t} := (t_i)_1^{n+k}$ with

$$t_i < t_{i+k}, \qquad \text{all} \quad i,$$

denote by $\mathbb{S}_{k,\mathbf{t}}$ the linear span of the n normalized B-splines $N_{1,k},..., N_{n,k}$, given by the rule that, for each t,

$$N_{i,k}(t) := g_k(t_i,..., t_{i+k}; t)(t_{i+k} - t_i),$$

the kth divided difference of

$$g_k(s; t) := (s - t)_+^{k-1}$$

as a function of s at the $k + 1$ points $t_i,..., t_{i+k}$. The elements of $\mathbb{S}_{k,\mathbf{t}}$ are called *polynomial splines of order k with knot sequence* $\mathbf{t}$.

Let $\boldsymbol{\tau} := (\tau_i)_1^n$ be a strictly increasing real sequence. As is shown in [12], *there exists, for given f, exactly one* $s \in \mathbb{S}_{k,\mathbf{t}}$ *such that*

$$s(\tau_i) = f(\tau_i), \qquad i = 1,..., n,$$

if and only if

$$N_{i,k}(\tau_i) \neq 0, \qquad i = 1,..., n,$$

i.e., *if and only if*

$$t_i < \tau_i < t_{i+k}, \qquad i = 1,..., n. \tag{1}$$

Hence, assuming $\boldsymbol{\tau}$ to satisfy (1), the conditions

$$Pf \in \mathbb{S}_{k,\mathbf{t}}, \qquad (Pf)(\tau_i) = f(\tau_i), \qquad i = 1,..., n, \tag{2}$$

* Sponsored by the United States Army under Contract No. DA-31-124-ARO-D-462.

define a linear map into $\mathbb{S}_{k,\mathbf{t}}$ which reproduces $\mathbb{S}_{k,\mathbf{t}}$. This paper is concerned with bounding P as a map on $C[t_1, t_{n+k}]$, i.e., with estimating

$$\| P \| := \sup_f \frac{\| Pf \|_\infty}{\| f \|_\infty},$$

where the sup is taken over all $f \in C[t_1, t_{n+k}]$, and

$$\| f \|_\infty := \sup_{t_1 \leqslant t \leqslant t_{n+k}} | f(t) |.$$

1. An Upper Bound

Since Pf depends only on the n-vector $(f(\tau_i))_1^n$ and since, given any n-vector $(f_i)_1^n$, there exists $f \in C[t_1, t_{n+k}]$ such that

$$f(\tau_i) = f_i, \qquad i = 1,..., n, \qquad \text{while} \quad \| f \|_\infty = \|(f_i)\|_\infty,$$

it follows that

$$\| P \| = \sup_f \| Pf \|_\infty / \| f \|_\infty = \sup_f \| Pf \|_\infty / \|(f(\tau_i))\|_\infty .$$

But then, since $(Pf)(\tau_i) = f(\tau_i)$, $i = 1,..., n$, while ran $P = \mathbb{S}_{k,\mathbf{t}}$, it follows that

$$\| P \| = \sup_{s \in \mathbb{S}_{k,\mathbf{t}}} \| s \|_\infty / \|(s(\tau_i))_1^n \|_\infty .$$

Writing the general element s of $\mathbb{S}_{k,\mathbf{t}}$ in terms of its B-spline representation, this gives that

$$\| P \| = \sup_{\mathbf{a}} \Big\| \sum_j a_j N_{j,k} \Big\|_\infty \Big/ \max_i \Big| \sum_j a_j N_{j,k}(\tau_i) \Big|.$$

By [1], there exists a positive D_k depending only on k such that

$$D_k^{-1} \| \mathbf{a} \|_\infty \leqslant \Big\| \sum_j a_j N_{j,k} \Big\|_\infty \leqslant \| \mathbf{a} \|_\infty, \qquad \text{all } \mathbf{a} \in \mathbb{R}^n.$$

Since

$$\sup_{\mathbf{a}} \| \mathbf{a} \|_\infty \Big/ \max_i \Big| \sum_j a_j N_{j,k}(\tau_i) \Big| = \|(N_{j,k}(\tau_i))^{-1} \|_\infty ,$$

we therefore obtain the estimate

$$D_k^{-1} \| G^{-1} \|_\infty \leqslant \| P \|_\infty \leqslant \| G^{-1} \|_\infty ,$$

showing that bounding P in the uniform norm is equivalent to bounding below the $n \times n$ matrix

$$G := (N_{j,k}(\tau_i))_{i,j=1}^{n} \tag{1}$$

with respect to the matrix norm associated with the max-norm for vectors. This proves the following.

LEMMA 1.1. *As a linear map on* $C[t_1, t_{n+k}]$, *the map* P *of spline interpolation given by* (0.2) *satisfies*

$$D_k^{-1} \| G^{-1} \|_\infty \leqslant \| P \|_\infty \leqslant \| G^{-1} \|_\infty$$

for some positive constant D_k *depending only on* k, *and with* G *the Gramian matrix* (1).

Finding a lower bound for a matrix, i.e., an upper bound for its inverse, is in general very difficult. In this particular case, one would expect some help from the fact that G *is totally nonnegative*, i.e., G has all its minors nonnegative, as is shown in [6; Chap. 10, Theorem 4.1]. But the only use I have been able to make of this fact has been in the form of its simple consequence that all $(n-1)$-minors of G are nonnegative, hence G^{-1} is a checkerboard matrix. This implies (see [2; Lemma 2.4])

LEMMA 1.2. *Let* $D := ((-1)^i \delta_{ij})$, *with* δ_{ij} *the Kronecker delta. If, for some* n*-vector* γ,

$$\min_{1 \leqslant i \leqslant n} (DGD^{-1}\gamma)(i) > 0,$$

then

$$\| \gamma \|_\infty / \max_i (DGD^{-1}\gamma)(i) \leqslant \| G^{-1} \|_\infty \leqslant \| \gamma \|_\infty / \min_i (DGD^{-1}\gamma)(i).$$

For given n-vector γ, set

$$\mathbf{a} := D^{-1}\gamma.$$

Then $\| \mathbf{a} \|_\infty = \| \gamma \|_\infty$, while

$$(DGD^{-1}\gamma)(i) = (DG\mathbf{a})(i) = (-)^i \sum_j a_j N_{j,k}(\tau_i).$$

Lemmas 1.1 and 1.2 have therefore the following Corollary.

COROLLARY. *If $s \in \mathbb{S}_{k,\mathbf{t}}$ has B-spline coefficients* $\mathbf{a}$ *and satisfies*

$$s(\tau_i)\, s(\tau_{i+1}) < 0, \qquad i = 1,\dots, n-1,$$

then

$$\| P \| \leqslant \| \mathbf{a} \|_\infty / \min_i | s(\tau_i)|.$$

Finally, from [4], if $s = \sum_j a_j N_{j,k}$ and $\tau_i \in (t_i, t_{i+k})$ (as we assume) then

$$a_i = s(\tau_i) + \sum_{j=1}^{k-1} (-)^{k-1-j}\, \psi_{ik}^{(k-1-j)}(\tau_i)\, s^{(j)}(\tau_i)$$

with

$$\psi_{ik}(t) := (t_{i+1} - t) \cdots (t_{i+k-1} - t)/(k-1)!,$$

hence

$$a_i - \sum_{j=1}^{k-1} (-)^{k-1-j}\, \psi_{ik}^{(k-1-j)}(\tau_i)\, s^{(j)}(\tau_i) = s(\tau_i), \tag{2}$$

which is of help in relating the vectors $\mathbf{a}$ and $(s(\tau_i))$ or, equivalently, in computing the entries of G.

2. A Lower Bound

There is no hope of bounding P independently of $\boldsymbol{\tau}$. For one, one would expect $\| P \|$ to blow up as $\boldsymbol{\tau}$ approaches a sequence violating (0.1). For another, $\| P \|$ is guaranteed to approach infinity as two consecutive interpolation points approach each other ($\mathbf{t}$ being held fixed). For, in this situation, the interpolation process approaches osculatory interpolation at the limit of the two interpolation points. But such a process cannot be bounded in the sup-norm since derivative evaluation cannot be bounded in the sup-norm.

In order to make this last argument precise, and for further guidance, I prove the following.

LEMMA. *Let r be a positive integer less than k and assume, for simplicity, that*

$$t_i < t_{i+k-r}, \qquad \textit{all} \quad i.$$

Then, for $i = 1,\dots, n + r$,

$$\| P^{(r-1)} \| := \sup_f \frac{\|(Pf)^{(r-1)}\|_\infty}{\| f^{(r-1)} \|_\infty} \geqslant \mathrm{const}_{k,r}\, d_{i,r}/(\tau_{i+r} - \tau_i)$$

with

$$d_{i,r} := \min\{t_{j+k-r} - t_j \mid (t_j, t_{j+k-r}) \cap (\tau_i, \tau_{i+r}) \neq \varnothing\}.$$

Proof. Let $\mathbf{a}$ be the coordinate vector for $(Pf)^{(r-1)} \in \mathbb{S}_{k-r+1,\mathbf{t}}$ with respect to the B-spline basis. By [1], there exists $D_{k-r+1} > 0$ depending only on $k - r + 1$ so that

$$D_{k-r+1}^{-1} \| \mathbf{a} \|_\infty \leqslant \|(Pf)^{(r-1)} \|_\infty \leqslant \| \mathbf{a} \|_\infty ,$$

while (see, e.g., [3; (14)–(15)])

$$(Pf)^{(r)} = (k - r) \sum_{i=1}^{n+r} \frac{a_i - a_{i-1}}{t_{i+k-r} - t_i} N_{i,k-r} .$$

Hence, for $\tau_i \leqslant t \leqslant \tau_{i+r}$,

$$\begin{aligned} |(Pf)^{(r)}(t)| &\leqslant (k - r) \max \left\{ \frac{a_j - a_{j-1}}{t_{j+k-r} - t_j} \,\middle|\, N_{j,k-r}(t) \neq 0 \right\} \\ &\leqslant (k - r)\, 2 \| \mathbf{a} \|_\infty / d_{i,r} . \end{aligned}$$

But then

$$\begin{aligned} | f[\tau_i ,..., \tau_{i+r}]| &= |(Pf)[\tau_i ,..., \tau_{i+r}]| \\ &\leqslant \sup_{\tau_i \leqslant t \leqslant \tau_{i+r}} |(Pf)^{(r)} (t)|/r! \\ &\leqslant (k - r)\, 2D_{k-r+1} \|(Pf)^{(r-1)} \|_\infty/(r!\, d_{i,r}). \end{aligned}$$

Therefore,

$$\sup_f \|(Pf)^{(r-1)} \|_\infty/\| f^{(r-1)} \|_\infty \geqslant \frac{d_{i,r} r!}{(k - r)\, 2D_{k-r+1}} \sup_f \frac{| f[\tau_i ,..., \tau_{i+r}]|}{\| f^{(r-1)} \|_\infty} .$$

But this last supremum can be shown to be at least

$$\frac{2}{r!\, (\tau_{i+r} - \tau_i)}$$

which is obvious for $r = 1$ and is proved for $r > 1$ as follows:

$$\begin{aligned} f[\tau_i ,..., \tau_{i+r}] &= \int_{\tau_i}^{\tau_{i+r}} g_r(\tau_i ,..., \tau_{i+r} ; s) f^{(r)}(s)\, ds/(r - 1)! \\ &= - \int_{\tau_i}^{\tau_{i+r}} (d/ds)\, g_r(\tau_i ,..., \tau_{i+r} ; s) f^{(r-1)}(s)\, ds/(r - 1)! \end{aligned}$$

since $g_r(\tau_i ,..., \tau_{i+r} ; \cdot)$ is a B-spline of order r with knots $\tau_i ,..., \tau_{i+r}$ (see Section 0), hence vanishes at τ_i and τ_{i+r}. Further [5; Theorem 1],

$$(d/ds)\, g_r(\tau_i ,..., \tau_{i+r}; .)$$

changes sign exactly once in $[\tau_i, \tau_{i+r}]$, at t^* such that

$$\max_{\tau_i \leqslant s \leqslant \tau_{i+r}} g_r(\tau_i, \ldots, \tau_{i+r}; s) = g_r(\tau_i, \ldots, \tau_{i+r}; t^*).$$

Hence,

$$\int_{\tau_i}^{\tau_{i+r}} |(d/ds)\, g_r(\tau_i, \ldots, \tau_{i+r}; s)|\, ds = 2g_r(\tau_i, \ldots, \tau_{i+r}; t^*).$$

But since

$$\int_{\tau_i}^{\tau_{i+r}} g_r(\tau_i, \ldots, \tau_{i+r}; s)\, ds/(r-1)! = 1/r!$$

we must have

$$\max_{\tau_i \leqslant s \leqslant \tau_{i+r}} g_r(\tau_i, \ldots, \tau_{i+r}; s) \geqslant \frac{1}{r(\tau_{i+r} - \tau_i)}.$$

Consequently,

$$\sup_f \frac{f[\tau_i, \ldots, \tau_{i+r}]}{\| f^{(r-1)} \|_\infty} = \int_{\tau_i}^{\tau_{i+r}} |(d/ds)\, g_r(\tau_i, \ldots, \tau_{i+r}; s)|\, ds/(r-1)!$$
$$\geqslant \frac{2}{r!\,(\tau_{i+r} - \tau_i)},$$

and the asserted lower bound for $\| P^{(r-1)} \|$ follows with

$$\text{const}_{k,r} := (k - r)^{-1} D_{k-r+1}^{-1}. \qquad \text{Q.E.D.}$$

The case $r = 1$ of this lemma shows that there is no hope of bounding P unless $\boldsymbol{\tau}$ is tied very closely to $\mathbf{t}$ in such a way that $\Delta\tau_i$ being "small" implies that $t_{j+k-1} - t_j$ is "small" for some t_j "near" τ_i. Consider, in particular, *odd-degree spline interpolation at knots* (without the use of boundary derivatives), i.e.,

$$k = 2m$$

for some $m \in \mathbb{N}$, and

$$\tau_{m+i} = t_{k+i}, \qquad i = 1, \ldots, n-k,$$

while the first m τ_i's are chosen in $[t_k, t_{k+1})$ and the last m τ_i's are, similarly, chosen in $(t_n, t_{n+1}]$. If $k > 2$, then we can make the norm of this process arbitrarily large (even for fixed n) merely by letting two consecutive knots (and interpolation points), τ_i and τ_{i+1} say, approach each other. For, this will decrease $\tau_{i+1} - \tau_i$ to zero while not materially decreasing $t_{j+k-1} - t_j$ for any j. Note that Nord's example [11] shows only the unboundedness of cubic spline interpolation as n approaches infinity.

If the knot sequence $\mathbf{t}$ satisfies

$$t_i < t_{i+k-r}\,, \qquad \text{all} \quad i,$$

for a given integer r, then, by restricting P to $C^{(r-1)}[t_1\,, t_{n+k}]$, we can consider the map $P^{(r-1)}$ which associates $f^{(r-1)} \in C[t_1\,, t_{n+k}]$ with $(Pf)^{(r-1)} \in \mathbb{S}_{k-r,\mathbf{t}}$, i.e.,

$$P^{(r-1)} f^{(r-1)} = (Pf)^{(r-1)}, \qquad \text{all} \quad f^{(r-1)} \in C[t_1\,, t_{n+k}].$$

The lemma shows that $P^{(r-1)}$ cannot be bounded in the sup-norm unless

$$\max_i \min \left\{ \frac{t_{j+k-r} - t_j}{\tau_{i+r} - \tau_i} \,\middle|\, (t_j\,, t_{j+k-r}) \cap (\tau_i\,, \tau_{i+r}) \neq \varnothing \right\} \tag{1}$$

can be bounded. For the case of odd-degree spline interpolation at knots mentioned before, this means that $P^{(r-1)}$ cannot be bounded in the sup-norm *independently of* $\mathbf{t}$ unless $k - r \leqslant r$. Since, for reasons to be given elsewhere, $P^{(s)}$ cannot be bounded in the sup-norm independently of $\mathbf{t}$ for $s > m$, this leaves $P^{(m-1)}$ and $P^{(m)}$ as the only candidates. In the case $k = 4$ of cubic spline interpolation at knots, i.e., when $m = 2$, these two are indeed known to be bounded independently of $\mathbf{t}$ (as can be deduced from [13]). For $k = 6$, $P^{(m)}$ has been shown to be bounded independently of $\mathbf{t}$ in [2]. But the question of bounding $P^{(k/2)}$ or $P^{(k/2-1)}$ for arbitrary (even) k is still wide open.

Finally, we note that the lower bound given in the lemma is far from strict. For, this bound can be bounded above in terms of the local mesh ratio whereas, e.g., in cubic spline interpolation at knots, P is known [8] not to be boundable in terms of the *local* mesh ratio.

3. An Upper Bound for Cubic Spline Interpolation at Knot Averages

In [9], Marsden treats in detail the case $k = 3$ of quadratic spline interpolation. He shows that, *with the choice*

$$\tau_i = (t_{i+1} + t_{i+2})/2, \qquad i = 1,..., n, \tag{1}$$

$\| P \|$ *is bounded by* 2 *regardless of* $\mathbf{t}$, surely a remarkable result. Marsden further conjectures that in the case $k = 4$ of cubic spline interpolation, the choice

$$\tau_i = (t_{i+1} + t_{i+2} + t_{i+3})/3, \qquad i = 1,..., n,$$

results in P which can be bounded in the sup-norm independently of $\mathbf{t}$.

Marsden was apparently led to this particular choice because of his joint work with Schoenberg [10], [7], in which the very simple linear map V, given by the rule

$$Vf := \sum_i f(\tau_{i,k})\, N_{i,k}$$

with

$$\tau_{i,k} = (t_{i+1} + \cdots + t_{i+k-1})/(k-1), \qquad i = 1,\dots, n,$$

is shown to be variation diminishing. Note that, for this choice of τ_i's and for $r = 1$, the quantity in (2.1) becomes $k - 1$ since $\tau_{i+1} - \tau_i = (t_{i+k} - t_{i+1})/(k-1)$. This means that, for this choice, the lemma in Section 2 produces the lower bound D_{k-1}^{-1} for $\| P \|$.

Before proving Marsden's conjecture with the aid of the corollary in Section 1, I want to derive in that way a bound for parabolic spline interpolation at knot averages in order to illustrate the procedure.

Choosing $k = 3$ and Marsden's interpolation nodes (1), (1.2) becomes

$$a_i + \tfrac{1}{8}(\Delta t_{i+1})^2\, s''(\tau_i) = s(\tau_i),$$

or, with

$$s''(t) = 2 \sum_j \left(\frac{\Delta a_{j-1}}{t_{j+2} - t_j} - \frac{\Delta a_{j-2}}{t_{j+1} - t_{j-1}}\right)\Big/ \Delta t_j\, N_{j,1}(t)$$

(see [3; (14)–(15)]),

$$a_i + \Delta t_{i+1} \left(\frac{\Delta a_i}{t_{i+3} - t_{i+1}} - \frac{\Delta a_{i-1}}{t_{i+2} - t_i}\right)\Big/ 4 = s(\tau_i)$$

or

$$\frac{1}{4} \frac{\Delta t_{i+1}}{t_{i+2} - t_i}\, a_{i-1} + \left(1 - \frac{\Delta t_{i+1}}{4} \left(\frac{1}{t_{i+2} - t_i} + \frac{1}{t_{i+3} - t_{i+1}}\right)\right) a_i$$
$$+ \frac{1}{4} \frac{\Delta t_{i+1}}{t_{i+3} - t_{i+1}}\, a_{i+1} = s(\tau_i)$$

which shows G to be tridiagonal and column diagonally dominant, but, unfortunately for us, *not* necessarily row diagonally dominant. In the terms of the corollary in Section 1, this means that the simple choice

$$a_i = (-1)^i, \qquad \text{all} \quad i,$$

will not give $s(\tau_i)\, s(\tau_{i+1}) < 0$ for all i and all $\mathbf{t}$.

Consider now the choice

$$a_i = (-)^i \left(1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i}\right), \quad \text{all } i, \tag{2}$$

so that

$$\| \mathbf{a} \|_\infty \leqslant 2.$$

Then

$$\begin{aligned}
(-)^i s(\tau_i) &\geqslant -\frac{1}{4} \frac{\Delta t_{i+1}}{t_{i+2} - t_i} \left(1 + \frac{\Delta t_i}{t_{i+2} - t_i}\right) \\
&\quad + \left(1 - \frac{\Delta t_{i+1}}{4} \left(\frac{1}{t_{i+2} - t_i} + \frac{1}{t_{i+3} - t_{i+1}}\right)\right)\left(1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i}\right) \\
&\quad - \frac{1}{4} \frac{\Delta t_{i+1}}{t_{i+3} - t_{i+1}} \left(1 + \frac{\Delta t_{i+2}}{t_{i+3} - t_{i+1}}\right) \\
&= 1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i} \\
&\quad - \frac{\Delta t_{i+1}}{4(t_{i+2} - t_i)} \left(1 + \frac{\Delta t_i}{t_{i+2} - t_i} + 1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i}\right) \\
&\quad - \frac{\Delta t_{i+1}}{4(t_{i+3} - t_{i+1})} \left(1 + \frac{\Delta t_{i+2}}{t_{i+3} - t_{i+1}} + 1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i}\right) \\
&\geqslant 1 + \frac{\Delta t_{i+1}}{t_{i+3} - t_i} - \frac{3}{4} \Delta t_{i+1} \left(\frac{1}{t_{i+2} - t_i} + \frac{1}{t_{i+3} - t_{i+1}}\right) \\
&=: f(A, B)
\end{aligned}$$

where, with

$$A := \Delta t_i / \Delta t_{i+1}, \qquad B := \Delta t_{i+2} / \Delta t_{i+1},$$

$$f(A, B) = 1 + \frac{1}{A + B + 1} - \frac{3}{4} \left(\frac{1}{1 + A} + \frac{1}{1 + B}\right).$$

But, for nonnegative A, B,

$$\begin{aligned}
f(A, B) &= \frac{A + B + 2}{A + B + 1} - \frac{3}{4} \frac{A + B + 2}{(A + 1)(B + 1)} \\
&= \frac{A + B + 2}{A + B + 1} \left(1 - \frac{3}{4} \frac{A + B + 1}{(A + 1)(B + 1)}\right) \\
&\geqslant \frac{A + B + 2}{A + B + 1} \frac{1}{4} \geqslant \frac{1}{4},
\end{aligned}$$

since, for $AB \geqslant 0$, $A + B + 1 \leqslant (A+1)(B+1)$. Note that $f(0, \infty) = \frac{1}{4}$, hence the lower bound of $\frac{1}{4}$ for $f(A, B)$ on $A \geqslant 0$, $B \geqslant 0$ is sharp.

In conclusion, for the choice (2) for $\mathbf{a}$,

$$(-)^i\, s(\tau_i) \geqslant \tfrac{1}{4}, \qquad \text{while} \quad \|\mathbf{a}\|_\infty \leqslant 2;$$

hence, from the corollary in Section 1,

$$\|P\| \leqslant 8$$

in this case. This should be compared with Marsden's result that

$$\|P\| \leqslant 2.$$

Now for the main point of this paper.

THEOREM. *Let P be the linear map of interpolation by elements of $\mathbb{S}_{4,\mathbf{t}}$ at the points of $\boldsymbol{\tau} = (\tau_i)_1^n$. If*

$$\tau_i = (t_{i+1} + t_{i+2} + t_{i+3})/3, \qquad i = 1,\ldots, n,$$

then

$$\|P\| \leqslant 27.$$

Proof. For this choice of $k = 4$ and the specific τ_i's, (1.2) becomes

$$\begin{aligned} a_i - \tfrac{1}{6}\{(t_{i+1} - \tau_i)(t_{i+2} - \tau_i) + (t_{i+1} - \tau_i)(t_{i+3} - \tau_i) \\ + (t_{i+2} - \tau_i)(t_{i+3} - \tau_i)\}\, s''(\tau_i) \\ - \tfrac{1}{6}(t_{i+1} - \tau_i)(t_{i+2} - \tau_i)(t_{i+3} - \tau_i)\, s'''(\tau_i) = s(\tau_i). \end{aligned} \tag{3}$$

With the abbreviation

$$a_j^{(2)} := \left(\frac{\Delta a_{j-1}}{t_{j+3} - t_j} - \frac{\Delta a_{j-2}}{t_{j+2} - t_{j-1}}\right)\Big/(t_{j+2} - t_j),$$

we have

$$s'' = 6 \sum_j a_j^{(2)} N_{j,2} \qquad \text{and} \qquad s''' = 6 \sum_j \frac{\Delta a_{j-1}^{(2)}}{\Delta t_j} N_{j,1}\,.$$

Hence, assuming without loss that

$$\Delta t_{i+2} \leqslant \Delta t_{i+1}\,, \qquad \text{therefore} \quad \tau_i \in [t_{i+1}\,, t_{i+2}], \tag{4}$$

we find that

$$\begin{aligned} s''(\tau_i) &= 6a_i^{(2)} N_{i,2}(\tau_i) + 6a_{i+1}^{(2)} N_{i+1,2}(\tau_i), \\ s'''(\tau_i) &= 6\Delta a_i^{(2)}/\Delta t_{i+1}\,. \end{aligned}$$

On substituting this into (3) and simplifying, we obtain

$$a_i + a_i^{(2)}(t_{i+2} - \tau_i)^3/\Delta t_{i+1} + a_{i+1}^{(2)}(\tau_i - t_{i+1})^3/\Delta t_{i+1} = s(\tau_i).$$

Now set

$$a_j = (-)^j \gamma_j, \qquad \text{all} \quad j,$$

for some *positive* γ still to be determined. Then we obtain, more explicitly, that

$$(-)^i s(\tau_i) = \gamma_i + \left(\frac{\gamma_i + \gamma_{i-1}}{t_{i+3} - t_i} + \frac{\gamma_{i-1} + \gamma_{i-2}}{t_{i+2} - t_{i-1}}\right) \frac{(t_{i+2} - \tau_i)^3}{t_{i+2} - t_i} \Big/ \Delta t_{i+1}$$

$$- \left(\frac{\gamma_{i+1} + \gamma_i}{t_{i+4} - t_{i+1}} + \frac{\gamma_i + \gamma_{i-1}}{t_{i+3} - t_i}\right) \frac{(\tau_i - t_{i+1})^3}{t_{i+3} - t_{i+1}} \Big/ \Delta t_{i+1}, \tag{5}$$

an expression to be bounded below by some positive quantity. In fact, *with the choice*

$$\gamma_j = \left(1 + \frac{t_{j+3} - t_{j+1}}{t_{j+4} - t_j}\right), \quad \textit{all } j,$$

we have $\| \mathbf{a} \|_\infty \leqslant 2$, *and*

$$(-)^i s(\tau_i) \geqslant 2/27, \qquad \textit{all} \quad i, \tag{6}$$

which, by the corollary in Section 1, finishes the proof of the theorem.

Unfortunately, I have been unable to come up with an elegant proof of the inequality (6). Instead, I had to follow the following procedure:

(i) Replace the term $(\gamma_{i-1} + \gamma_{i-2})/(t_{i+2} - t_{i-1})$ in the right hand side of (5) by zero, thereby obtaining a lower bound for $(-)^i s(\tau_i)$.

(ii) Multiply the resulting expression by 27, then subtract 2. It remains to prove the resulting expression nonnegative.

(iii) In the resulting expression, use the abbreviations

$$A := \Delta t_i/\Delta t_{i+1}, \qquad C := \Delta t_{i+2}/\Delta t_{i+1}, \qquad D := \Delta t_{i+3}/\Delta t_{i+1},$$
$$E := 1 + C = (t_{i+3} - t_{i+1})/\Delta t_{i+1}$$

bringing it into the form

$$27\gamma_i + (\gamma_i + \gamma_{i-1}) \operatorname{coef}_{i-1} + (\gamma_i + \gamma_{i+1}) \operatorname{coef}_{i+1} - 2 \tag{7}$$

with

$$\mathrm{coef}_{i-1} = \left(\frac{(2-E)^3}{1+A} - \frac{(1+E)^3}{E}\right)\Big/(A+E)$$

$$\mathrm{coef}_{i+1} = -\frac{(1+E)^3}{E}\Big/(E+D).$$

Note that our earlier assumption (4) translates into

$$1 \leqslant E \leqslant 2 \tag{8}$$

hence (7) has to be shown to be nonnegative on $A, D \geqslant 0$, $1 \leqslant E \leqslant 2$.

(iv) Verify that both coef_{i-1} and coef_{i+1} are always negative, hence replace without increase in (7)

$\gamma_i + \gamma_{i-1}$ by the larger expression $3 + 1/(A + E + D)$

and

$\gamma_i + \gamma_{i+1}$ by the larger expression $3 + C/(A + E + D)$.

(v) Bring the resulting expression on one denominator, using the further abbreviation

$$F := A + D.$$

The denominator is then positive on $A, D \geqslant 0$, $1 \leqslant E \leqslant 2$, while the numerator is a polynomial in A, D, E, and F whose coefficients are given in the following table:

	1	A	D	AD	A^2	DA^2	D^2	$(1+A)F^2$	$(1+A)ADF$
E^6	−3								
E^5	10	−10	−6						
E^4	1	39	26	−12	−10		−3		
E^3	−9	−9	−17	67	48		18	−3	
E^2	1	−10	6	−14	−27	52	−36	16	
E		−7	−4	12	−7		24	−9	25
1		1	−1	−1	1			−3	

(vi) Verify that, on $1 \leqslant E \leqslant 2$, each of the 9 polynomials in E listed in the above table is nonnegative. As it turns out, all of these polynomials are strictly positive on [1, 2] except for the one multiplying D^2, the polynomial

$$E(24 - 36E + 18E^2 - 3E^3)$$

which decreases from a value of 3 at 1 monotonely to a value of 0 at 2.

It follows that, for A, D, and F nonnegative, the numerator is nonnegative, thus proving (6); Q.E.D.

In view of the fact that I obtained above a bound of 8 in the parabolic case when actually the norm is at most 2, it seems likely that, in the cubic case, the norm is at most 3 or 4 rather than the proven 27.

At this point, it is easy to conjecture that kth order spline interpolation at the averages of $k - 1$ successive knots is bounded as a map on $C[t_1, t_{n+k}]$ independently of $\mathbf{t}$. It is also easy to see that a proof of this conjecture is not likely to be a straightforward generalization of the above procedure.

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